ON THE EQUATIONS OF THE PRECESSIONAL THEORY OF A GYROSCOPE IN THE FORM OF EQUATIONS OF MOTION OF THE POLE IN THE PHASE PLANE

(OB URAVNENIIAKH PRETSESSIONNOI TEORII GIROSCOPOV V FORME URAVNENII DVIZHENIIA IZOBRAZHAIUSHCHEI TOCHKI V KARTINNOI PLOSKOSTI)

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This paper gives a rigorous justification of the equations of motion of a gyroscope in the form of equations of the representative point in a certain plane, the so-called phase plane. It turns out that these well-known equations [1,2,3] are valid only if the forces acting on the gyroscope satisfy a number of limitations.

1. The representative point (the pole) of a gyroscope and the phase plane are introduced in the following way. Let xyz be a coordinate system whose origin coincides with the center of the gimbal suspension, the z-axis making a small angle with the z-axis which is the spin axis of the rotor (or equivalently with the direction of the angular momentum vector H of the rotor) (see Fig. 1). The orientation of the coordinate system xyz at any instant of time is assumed to be known. In particular, ω_x, ω_y and ω_z which are the x, y and z components of the angular velocity of the system xyz with respect to the system $\xi^*\eta^*\zeta^*$; which has constant orientation with respect to fixed stars, are known functions of the time. The directions of the axes x and y are chosen in such a way that the components ω_x and ω_y assume the simplest possible form. For example, in the case of a vertical gyroscope it is convenient in many cases to select the geographical system, where the z-axis coincides with the local vertical and the x- and y-axes are directed East and North, respectively. In the case of a gyrocompass the z-axis would point North. In a number of theoretical problems (for example, for a horizontal gyrocompass, see [4]), it is convenient to select a coordinate system x^0, y^0, z^0, in which the x^0-axis is along the velocity vector of the suspension point of the gyroscopic system with respect to a hypothetical non-rotating earth. The z^0-axis is taken along the radius vector of this sphere.

The plane XY parallel to the coordinate plane xy (Fig. 1) at a unit distance from the origin (its equation is $z = 1$) is called in the theory of gyroscopes the phase plane. The point P which is the intersection of the phase plane with the angular momentum vector H (or with the z'-axis) is called the representative point or the pole of the gyroscope. The x, y and z components of the velocity of the representative point with respect to the inertial system $\xi^*\eta^*\zeta^*$ (whose origin is also in the center of suspension of the gyroscope) are expressed by the following formulas:

$$v_x = \frac{dx}{dt} + \omega_y - y\omega_z \qquad v_y = \frac{dy}{dt} - \omega_x + x\omega_z, \qquad v_z = y\omega_x - x\omega_y \qquad (1)$$

Here x and y are the coordinates of the representative point P (the z-coordinate is constant and equals unity).

The equations of motion of the representative point for the precessional (elementary) theory of a gyroscope are usually expressed in the following form

$$Hv_x = M_x, \qquad Hv_y = M_y \qquad (2)$$

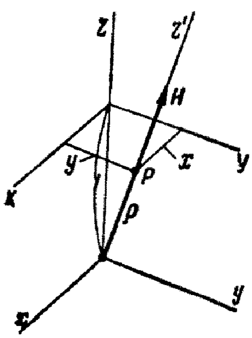

Fig. 1.

Strictly speaking, in these equations the right-hand members M_x and M_y should not be (as it often is) regarded as sums of the moments of forces acting on the gyroscope with respect to the x- and y-axis, respectively. A gyroscope is a mechanical system consisting of the following three parts: the rotor, the inner gimbal ring, and the outer gimbal ring (Fig. 2). For this reason, when we construct the equations of motion of a gyroscope we must carefully take into account on which part the appropriate forces act. Otherwise mistakes are unavoidable*.

In what follows we shall explain how to interpret the right-hand members of the equations (2), how to make the equations more accurate and we shall list the conditions under which the equations are valid.

2. In order to accomplish the tasks stated above we must use the rigorous equations for the precessional motion of a gyroscope on gimbals. As these equations will be derived from the principle of virtual velocities [5] we must first introduce certain kinematic relations.

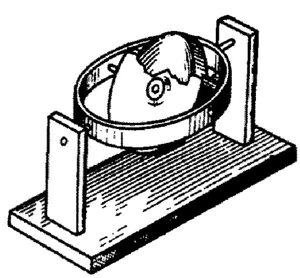

Fig. 2.

We shall introduce three new coordinate

* One such mistake can be found in [6].

systems $\xi\pi\zeta$; $\xi_1\pi_1\zeta_1$; and $x'y'z'$ with origins in the center of the gyroscope suspension, like the systems $\xi^*\eta^*\zeta^*$ and xyz (Fig. 3). The coordinate system $\xi\eta\ \zeta$ is attached to the base of the gyroscope system, the ξ-axis being along the axis of the outer gimbal ring. The coordinate system $\xi_1\eta_1\zeta_1$ is attached to the outer gimbal ring, the ξ_1- and η_1-axes being along the axis of the outer ring and along the axis of the inner ring, respectively.

Finally, the coordinate system $x'y'z'$ is fixed in the inner ring, the y'-axis coinciding with the η_1-axis and z'-axis along the axis of rotation of the gyroscope (Fig. 4), that is along the vector of its angular momentum H.

We shall denote further by a the angle between the outer ring and the base, by β the angle between the inner and the outer rings. We shall introduce the condition that when $a = 0$ the coordinate system $\xi_1\eta_1\zeta_1$ and $\xi\eta\zeta$ coincide, and when the angle a increases the outer ring turns anticlockwise with respect to the base when observed from the side of positive branches of the coinciding axes ξ and ξ_1. Similarly, when $\beta = 0$ the coordinate systems $x'y'z'$ and $\xi_1\eta_1\zeta_1$ coincide and when β increases the inner ring turns anticlockwise with respect to the outer ring when observed from the side of the positive branches of the axes y' and η_1.

The angular velocity of the base with respect to the system $\xi^*\eta^*\zeta^*$ (of constant orientation with respect to the fixed stars), shall be denoted by u and its ξ, η and ζ components by u_ξ, u_η and u_ζ..

It can be easily shown (Fig. 5) that ω_1, the angular velocity of the outer ring with respect to the system $\xi^*\eta^*\zeta^*$, has the following $\xi_1\eta_1\zeta_1$ is the system moving with the outer ring):

$$\omega_{\xi_1}{}^1 = u_\xi + \frac{d\alpha}{dt}, \qquad \omega_{\eta_1}{}^1 = u_\eta \cos\alpha + u_\zeta \sin\alpha,$$

$$\omega_{\zeta_1}{}^1 = -u_\eta \sin\alpha + u_\zeta \cos\alpha \tag{3}$$

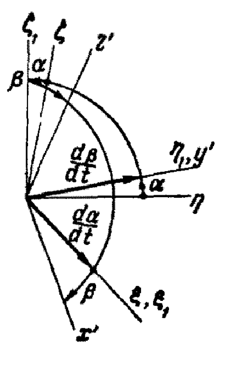

Fig. 3

In a similar way we can obtain the x', y' and z' components of ω', which is the angular velocity of the inner ring with respect to the system $\xi^*\eta^*\zeta^*$ (x', y', z' is the system moving with the inner ring):

$$\omega_{x'}' = \left(u_\xi + \frac{d\alpha}{dt}\right)\cos\beta - (-u_\eta \sin\alpha + u_\zeta \cos\alpha)\sin\beta$$

$$\omega_{y'}' = u_\eta \cos\alpha + u_\zeta \sin\alpha + \frac{d\beta}{dt} \tag{4}$$

$$\omega_{z'}' = \left(u_\xi + \frac{d\alpha}{dt}\right)\sin\beta + (-u_\eta \sin\alpha + u_\zeta \cos\alpha)\cos\beta$$

Finally, the x', y', z' components of ω, which is the angular velocity of the rotor with respect to the system $\xi^*\eta^*\zeta^*$ are

$$\omega_{x'} = \omega_{x'}', \qquad \omega_{y'} = \omega_{y'}', \qquad \omega_{z'} = \omega_{z'}' + \frac{d\varphi}{dt} \tag{5}$$

where ϕ is the angle of anticlockwise rotation of the rotor with respect to the inner ring.

3. We shall investigate the kinetostatics of the gyroscope in the system $\xi^*\eta^*\zeta^*$. As we have stated above the gyroscope is a mechanical system consisting of three bodies: outer gimbal ring, inner gimbal ring, and the rotor (Fig. 2).

In the precessional (elementary) theory of gyroscopes the angular momenta of the two rings are neglected, and the angular momentum of the rotor is assumed to be its natural angular momentum H. The vector H is directed along the rotor's axis of revolution z'. The absolute value of H equals the product of the rotor's moment of inertia about the z'-axis by the angular velocity of the rotor $d\phi/dt$.

It follows then, that according to the precessional theory of a gyroscope, the D'Alembert inertial forces* in the system $\xi^*\eta^*\zeta^*$ should be expressed only through variation of the proper angular momentum vector H.

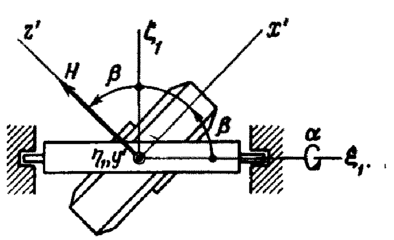

Fig. 4.

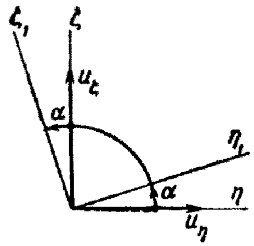

Fig. 5.

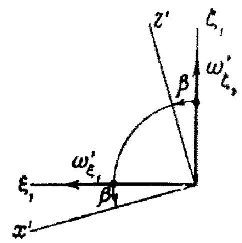

Fig. 6.

The velocity vector of the tip of the vector H (with respect to the $\xi^*\eta^*\zeta^*$ system) has the following $x'y'z'$ components (Fig. 7):

$$\omega_{y'}' H = -G_{x'}, \qquad -\omega_{x'}' H = -G_{y'}, \qquad \frac{dH}{dt} = -G_{z'} \tag{6}$$

Hence, when investigating the kinetostatics of a gyroscope mechanical system, we must use couples with moments $G_{x'}$, $G_{y'}$, $G_{z'}$ which are applied to the rotor and which are given by (6).

* An elementary D'Alembert inertial force is a vector in the direction opposite to the direction of acceleration (of a given element of mass) with respect to the considered coordinate system $\xi^*\eta^*\zeta^*$ and numerically equal to the product of the acceleration and the element of mass.

The forces acting on our system could be classified as the external forces and the mutual reaction forces between the rotor and the inner ring, between the inner and the outer ring, and between the outer ring and the base.

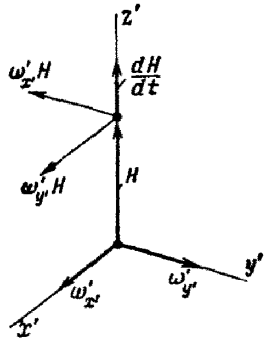

Fig., 7.

Among the external forces we include the Coriolis inertial forces and the inertial forces of the transfer motion caused by the displacement of the coordinate system $\xi^*\eta^*\zeta^*$ with respect to the so-called absolute system whose origin is at the center of mass of the solar system and having constant orientation with respect to fixed stars. The system $\xi^*\eta^*\zeta^*$ does not rotate, hence the Coriolis inertial forces are absent.

Due to translation of the coordinate system $\xi^*\eta^*\zeta^*$, all the inertial forces of the transfer motion acting on the separate elements of the mechanical system are mutually parallel. All of them have an opposite direction to the acceleration of the origin of $\xi^*\nu^*\zeta^*$ with respect to the absolute system. It is obvious that the totality of the inertial forces of the transfer motion acting on an isolated body is equivalent to a resultant force through the center of gravity. The magnitude of the resultant equals the product of the mass of the body and the acceleration of the origin of the system $\xi^*\eta^*\zeta^*$ (or of any other point in the body).

The forces of mutual reactions between different bodies in the gyroscope mechanical system consist of normal reactions of the constraints and of couples, whose vectors are along axes constraining these bodies. Such is also the case with the mutual reaction forces between the base and the outer gimbal ring. If there are some other forces of mutual reactions between the base and other bodies of the system they could be regarded as external forces.

On the strength of D'Alembert's principle, the gyroscope mechanical system under the action of all the listed forces together with the couples $G_{x'}$, $G_{y'}$, $G_{z'}$ should be in equilibrium. The system has three degrees of freedom: the motion of the base with respect to the system $\xi^*\eta^*\zeta^*$ should be regarded as known. The most natural choice of the generalized coordinates would be the set of angles α, β and ϕ, which are, respectively, the angles of rotation of the outer ring with respect to the base, of the inner ring with respect to the outer ring, and of the rotor with respect to the inner ring.

Let us impart to the elements of our mechanical system virtual velocities corresponding to the generalized virtual velocities

$$\delta(d\alpha/dt), \qquad \delta(d\beta/dt), \qquad \delta(d\varphi/dt)$$

The $\xi_1\eta_1\zeta_1$ projections of the virtual angular velocity of the outer ring, corresponding to the above virtual velocities are obtained from (3) as

$$\delta\omega_{\xi_1}^1 = \delta(d\alpha/dt), \qquad \delta\omega_{\eta_1}^1 = 0, \qquad \delta\omega_{\zeta_1}^1 = 0 \tag{7}$$

From formula (4), we obtain further the following expressions for the x', y', z' projections of the virtual angular velocity of the inner ring

$$\delta\omega_{x'}' = \delta\frac{d\alpha}{dt}\cos\beta, \qquad \delta\omega_{y'}' = \delta\frac{d\beta}{dt}, \qquad \delta\omega_{z'}' = \delta\frac{d\alpha}{dt}\sin\beta \tag{8}$$

Finally, we obtain the following expressions from formulas (5) and (4) for the projections of the virtual velocity of the rotor

$$\delta\omega_{x'} = \delta\frac{d\alpha}{dt}\cos\beta, \qquad \delta\omega_{y'} = \delta\frac{d\beta}{dt}, \qquad \delta\omega_{z'} = \delta\frac{d\alpha}{dt}\sin\beta + \delta\frac{d\varphi}{dt} \tag{9}$$

We shall construct now an expression for the virtual power δW due to all the forces acting on the gyroscope system, including the couples $G_{x'}$, $G_{y'}$, $G_{z'}$. We obtain

$$\delta W = (m_{x'} + G_{x'})\,\delta\omega_{x'} + (m_{y'} + G_{y'})\,\delta\omega_{y'} + (m_{z'} + G_{z'})\delta\omega_{z'} +$$
$$+ l_{x'}\,\delta\omega_{x'}' + l_{y'}\,\delta\omega_{y'}' + l_{z'}\,\delta\omega_{z'}' + k_{\xi_1}\delta\omega_{\xi_1}^1 + M_{z'}\delta\frac{d\varphi}{dt} + L_{y'}\delta\frac{d\beta}{dt} + K_{\xi_1}\delta\frac{d\alpha}{dt} \tag{10}$$

Here K_{ξ_1} is the sum of the moments of the forces exerted by the base on the outer ring about the axis of the outer ring $\xi_1(\xi)$; $L_{y'}$ is an analogous sum of the moments of the forces acting on the inner ring and exerted by the outer one, about the axis of the inner ring $y'(\eta_1)$; $M_{z'}$ is the sum of the moments of the forces acting on the rotor and exerted by the inner ring, about the rotor's rotation axis z'. Further, k, l and m, with appropriate subscripts, denote sums of moments of the outside forces acting respectively on the outer ring, inner ring and the rotor, about axes indicated by the subscripts.

In the expression for the virtual power δW the forces of normal reactions of constraints are obviously absent because they do not develop any power.

We shall now in the formula for δW replace the components of the virtual angular velocities by the expressions from (7), (8) and (9). We obtain

$$\delta W = [K_{\xi_1} + k_{\xi_1} + (l_{x'} + m_{x'} + G_{x'})\cos\beta + (l_{z'} + m_{z'} + G_{z'})\sin\beta]\,\delta\frac{d\alpha}{dt} +$$
$$+ (l_{y'} + m_{y'} + G_{y'} + L_{y'})\,\delta\frac{d\beta}{dt} + (m_{z'} + G_{z'} + M_{z'})\,\delta\frac{d\varphi}{dt} \tag{11}$$

According to the principle of virtual velocities, the power δW should equal zero for any choice of virtual velocities which satisfy the constraints, that is, for arbitrary values of $\delta(da/dt)$, $\delta(d\beta/dt)$ and $\delta(d\phi/dt)$. This is possible only if every multiplier of these velocities in the expression (11) equals zero. Consequently, we obtain the following relations

$$K_{\xi_1}+k_{\xi_1}+(l_{x'}+m_{x'}+G_{x'})\cos\beta+(l_{z'}+m_{z'}+G_{z'})\sin\beta=0$$
$$L_{y'}+l_{y'}+m_{y'}+G_{y'}=0,\qquad M_{z'}+m_{z'}+G_{z'}=0 \tag{12}$$

in which the quantities $G_{x'}$, $G_{y'}$ and $G_{z'}$, could be replaced by the expressions from (6). After eliminating a group of terms from the first expression of (12) we obtain

$$\omega'_{y'}H=m_{x'}+l_{x'}+(K_{\xi_1}+k_{\xi_1})\sec\beta-(M_{z'}-l_{z'})\operatorname{tg}\beta$$
$$-\omega'_{x'}H=m_{y'}+l_{y'}+L_{y'},\qquad dH/dt=m_{z'}+M_{z'} \tag{13}$$

The equations (13) are equations of the precessional motion of a gyroscope on gimbals.*

4. In order to pass now to the equations of motion of the representative point we shall begin by transforming the first two equations (13). The representative point is on the z'-axis at a variable distance ρ from the common origin of x', y', z' and xyz. It is easily seen that the x', y' and z' components of the velocity of the representative point with respect to the system $\xi^*\eta^*\zeta^*$ are given by the formulas

$$v_{x'}=\omega'_{y'}\rho,\qquad v_{y'}=-\omega'_{x'}\rho,\qquad v_{z'}=d\rho/dt \tag{14}$$

Using the above formulas we could write the first two equations (13) in the following form

$$H\frac{v_{x'}}{\rho}=m_{x'}+l_{x'}+(K_{\xi_1}+k_{\xi_1})\sec\beta-(M_{z'}-l_{z'})\operatorname{tg}\beta,$$
$$H\frac{v_{y'}}{\rho}=m_{y'}+l_{y'}+L_{y'} \tag{15}$$

The x', y' and z' components and the x, y and z components of the velocity of the representative point v are related through the equations

$$v_x=av_{x'}+bv_{y'}+cv_{z'}$$
$$v_y=a'v_{x'}+b'v_{y'}+c'v_{z'}$$
$$v_z=a''v_{x'}+b''v_{y'}+c''v_{z'} \tag{16}$$

* Compare the above derivation with the derivation by the method of elementary statics as given in [6].

The directions cosines between the coordinates systems x', y', z' and xyz are given in the tabular form in (17).

	x'	y'	z'
x	a	b	c
y	a'	b'	c'
z	a''	b''	c''

(17)

We shall substitute now in the right-hand members of (16) the expressions for $v_{x'}$, and $v_{y'}$ from (15) and the expression for $v_{z'}$ from the third formula in (14). Consequently we obtain

$$\begin{aligned} H\frac{v_x}{\rho} &= a(m_{x'} + l_{x'}) + b(m_{y'} + l_{y'}) + \\ &+ a[(K_{\xi_1} + k_{\xi_1})\sec\beta - (M_{z'} - l_{z'})\operatorname{tg}\beta] + bL_{y'} + \frac{cH}{\rho}\frac{d\rho}{at} \\ H\frac{v_y}{\rho} &= a'(m_{x'} + l_{x'}) + b''(m_{y'} + l_{y'}) + \\ &+ a'[(K_{\xi_1} + k_{\xi_1})\sec\beta - (M_{z'} - l_{z'})\operatorname{tg}\beta] + b'L_{y'} + \frac{c''H}{\rho}\frac{d\rho}{dt} \\ H\frac{v_z}{\rho} &= a''(m_{x'} + l_{x'}) + b\,(m_{y'} + l_{y'}) + \\ &+ a''[(K_{\xi_1} + k_{\xi_1})\sec\beta - (M_{z'} - l_{z'})\operatorname{tg}\beta] + b''L_{y'} + \frac{c''H}{\rho}\frac{d\rho}{dt} \end{aligned} \tag{18}$$

The third equation in (18) follows from the first two. This can be easily shown by adding separately the right- and the left-hand members of all these equations, multiplying them beforehand by the direction cosines c, c', c'' respectively, and utilizing the well-known relations between directions cosines (17), the equation

$$v_{z'} = cv_x + c'v_y + c''v_z \tag{19}$$

and the third formula in (14).

Let us mention that, similarly to the equations (16), we have

$$\begin{aligned} a(m_{x'} + l_{x'}) + b(m_{y'} + l_{y'}) + c(m_{z'} + l_{z'}) &= m_x + l_x \\ a'(m_{x'} + l_{x'}) + b'(m_{y'} + l_{y'}) + c'(m_{z'} + l_{z'}) &= m_y + l_y \\ a''(m_{x'} + l_{x'}) + b''(m_{y'} + l_{y'}) + c''(m_{z'} + l_{z'}) &= m_z + l_z \end{aligned} \tag{20}$$

where m_x, m_y, m_z, and l_x, l_y, l_z are sums of the moments of the external forces acting on the rotor and the inner ring about the indicated axes.

By using the equations (20) we could express the first two equations in (18) in the following form

$$H\frac{v_x}{\rho} = m_x + l_x + a\left[(K_{\xi_1} + k_{\xi_1})\sec\beta - (M_{z'} - l_{z'})\operatorname{tg}\beta\right] + \\ + bL_{y'} + c\left(H\frac{1}{\rho}\frac{d\rho}{dt} - m_{z'} - l_{z'}\right) \tag{21}$$

$$H\frac{v_y}{\rho} = m_y + l_y + a'\left[(K_{\xi_1} + k_{\xi_1})\sec\beta - (M_{z'} - l_{z'})\operatorname{tg}\beta\right] + \\ + b'L_{y'} + c'\left(H\frac{1}{\rho}\frac{d\rho}{dt} - m_{z'} - l_{z'}\right)$$

The above relations represent the rigorous equations of the motion of the representative point in the phase plane.

These equations should be supplemented by the equations relating direction cosines a, b, a' and b', which give the orientation of the moving object (that is, the coordinate system $\xi\eta\zeta$) with respect to both the coordinate system xyz and with respect to the coordinates of the representative point x and y which are in the xyz system. Besides, the two first formulas of (1) should be taken into account and also the formulas

$$x = \rho\cos z'x = \rho c, \qquad y = \rho\cos z'y = \rho c' \tag{22}$$

$$\rho = \sqrt{1 + x^2 + y^2} \tag{23}$$

5. The equations (21) could be considerably simplified in the case when the coordinates of the representative point x and y, and their time derivatives, are small quantities and their quadratic terms could be neglected. Then, according to the formula (23) the variable distance ρ becomes unity and the terms containing the time derivative of ρ vanish.

Besides, according to the formula (22), to small quantities of the second order we obtain

$$c = x, \qquad c' = y \tag{24}$$

Further, according to the table of direction cosines (17) we have

$$c^2 + c'^2 + c''^2 = 1, \qquad a''^2 + b''^2 + c''^2 = 1 \tag{25}$$

and with the same accuracy we have

$$c''^2 = 1 - x^2 - y^2, \quad c'' = 1 - \frac{x^2 + y^2}{2}, \quad a''^2 + b''^2 = x^2 + y^2 \tag{26}$$

From the above there follows that in the general case the direction cosines a'' and b'' are small quantities of the first order, and the direction cosine c'' differs from unity by a small quantity of a higher order. Taking into account the above observations and the relation

$$m_{z'} + l_{z'} = c\,(m_x + l_x) + c'\,(m_y + l_y) + c''\,(m_z + l_z) \tag{27}$$

we could, without changing the prescribed accuracy, replace in the equations (21) the sums of the moments $m_{z'}$, $l_{z'}$, by $m_z + l_z$ respectively.

Utilizing further the formulas (1) and (24), we finally obtain the differential equations of the motion for the representative point, in the form

$$\begin{aligned} H\left(\frac{dx}{dt} + \omega_y - y\omega_z\right) &= m_x + l_x - x\,(m_z + l_z) + \\ &+ a\,[(K_{\xi_1} + k_{\xi_1}) \sec\beta - (M_{z'} - l_{z'}) \operatorname{tg}\beta] + bL_{y'} \\ H\left(\frac{dy}{dt} - \omega_x + x\omega_z\right) &= m_y + l_y - y\,(m_z + l_z) + \\ &+ a'\,[(K_{\xi_1} + k_{\xi_1}) \sec\beta - (M_{z'} - l_{z'}) \operatorname{tg}\beta] + b'L_{z'} \end{aligned} \tag{28}$$

The direction cosines a, b, a' and b' between the axes x', y' and x, y, for small values of the coordinates x and y, could also be represented in a simpler form. They could be expressed explicitly through the angles α and β and the angle κ between the axes y' and y (the course of the object if the axis y is directed North).

6. We shall mention finally that the equations (2) are obtained from the rigorous equations (21) with the following additional assumptions: (1) The sum of the moments, K_ξ, of the forces of interaction between the base and the outer ring about the axis of the outer ring $\xi_1(\xi)$ is zero. In particular, this axis has no friction. Similarly, the moments $L_{y'}$ and $M_{z'}$ of the interaction forces of the inner ring with the outer ring and with the rotor are zero; (2) The sum of moments K_{ξ_1} of the external forces acting on the outer ring about the $\xi_1(\xi)$ axis, is zero (this is equivalent to the requirement that the outer ring should be exactly equilibrated); (3) The moments $m_{z'}$ and $l_{z'}$ of the external forces acting on the inner ring and the rotor, about the rotor's axis of rotation z' are zero. (4) The coordinates of the representative point x and y are small quantities.

Indeed, if we substitute in the equations (21)

$$K_{\xi_1} = L_{y'} = M_{z'} = 0, \qquad k_{\xi_1} = m_{z'} = l_{z'} = 0 \tag{29}$$

and further assume according to (23) that ρ equals unity, then we arrive at

$$Hv_x = m_x + l_x, \qquad Hv_y = m_y + l_y \tag{30}$$

which would be identical with equations (2) if we assume that M_x and M_y in these equations are sums of moments $m_k + l_x$ and $m_y + l_y$, of the external forces, acting on the mechanical system rotor - inner ring.

In this way we have fully clarified the conditions under which are valid the well-known equations (2) of the precessional theory of a gyroscope in the form of equations of motion of the representative point in the phase plane, and have also explained the meanings of individual terms of these equations.

It appears to be also possible to take into account the influence, on the motion of the axis of rotation of the gyroscope, of the moments K_{ξ_1}, $L_{y'}$, $M_{z'}$, k_{ξ_1}, $m_{z'}$, l_z, by using the equations in the form (21) or (28).

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